Global models’ formulation and associated transition matrices for the Rössler system, the electrodissolution of copper in phosphoric acid and the cycles of rainfed wheat observed from satellite in North Morocco.

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Two new algorithms *Polynomial Model Search* and *Global Modelling* were introduced in the main body of the present work [1]. Three systems are considered to test the robustness and the accuracy of both algorithms: one theoretical, one experimental and one environmental. The Rössler system [2] is used as theoretical benchmark. The three variables of this system were considered one by one in order to account for the different degrees of observability [3]. Measurements resulting from the electrodissolution of copper in phosphoric acid [4] were used as an experimental case. The cycle of rainfed wheat was chosen as an environmental case of study, based on vegetation index measured from satellite [5].

Global modeling aims at building mathematical models of concise description. Several models were obtained for each of the three systems considered in the study. These models are 3-dimensional (n = 3) and rely on Equation (3) presented in [1], in a polynomial formulation. The object of the present Supplemental Material is to provide explicitly the formulation of the models obtained for each of these systems and the transition matrices used for their validation (or invalidation). The transition matrices of the models are provided only when a partition could be obtained from the first return map. Transition matrices estimated from the original data set are also provided for comparison. The detailed introduction as well as a complete description of the data sets, theoretical background and algorithms are provided in [1] together with a discussion of the results.

1 The Rössler system

1.1 Rössler-x_2

Two models were obtained from the Rössler-x_2 variable, the 9-term model (of maximum degree q = 2) reads:

\[ P(X_1, X_2, X_3) = -4.571 \times 10^2 - 2.076 \times 10^1 X_3 \\
+ 3.855 \times 10^1 X_2 + 9.570 \times 10^1 X_2 X_3 - 3.127 \times 10^2 X_2^2 \\
- 9.350 \times 10^2 X_1 + 3.099 \times 10^2 X_1 X_3 + 4.760 \times 10^1 X_1 X_2 \\
- 1.217 \times 10^1 X_1^2 \] (1)

and the 7-term model (of maximum degree q = 2, also) reads:
\[ P(X_1, X_2, X_3) = -1.3824700 \cdot 10^3 X_3 + 6.927122 \cdot 10^{-4} X_2 X_3 \\
- 3.3915271 X_2^2 - 6.970505 \cdot 10^2 X_1 - 1.900433 X_1 X_3 \\
+ 3.439617 \cdot 10^1 X_1 X_2 - 9.389699 \cdot 10^1 X_1^2 \]

The (Markov) transfer matrix and corresponding binary transfer matrix were estimated from the first return maps presented in Figure 5 (see Tables 2 & 3 for details). For the 9-term model, these matrices read:

\[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0.28 & 0.23 & 0.23 & 0.27 \\
0 & 1 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0.07 & 0.05 & 0.06 & 0.07 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

\[ (1 1 1 1) \]

and for the 7-term model:

\[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0.14 & 0.36 & 0.36 & 0.14 \\
0 & 1 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0.16 & 0.06 & 0.07 & 0.01 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

\[ (1 1 1 1 1) \]

These matrices should be compared to the following ones estimated from the original signal of variable \( x_2 \):

\[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0.23 & 0.28 & 0.25 & 0.25 \\
0 & 1 & 2 & 3
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0.02 & 0.10 & 0.06 & 0.08 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

\[ (1 1 1 1 1) \]

\[ (1 1 1 1 0) \]

\[ (1 1 1 1 1) \]

1.2 Rössler-\( x_1 \)

Three models were obtained from the \( x_1 \) variable. The 13-term model (of maximum degree \( q = 4 \)) reads:
\[
P(X_1, X_2, X_3) = -1.218738.10^1 X_3 - 7.764914.10^3 X_3^2 \\
- 6.308658.10^5 X_2 X_3^2 - 5.553971.10^2 X_1 - 2.849993 X_1 X_3 \\
+ 7.498988.10^3 X_1 X_3 - 0.267307 X_1 X_2 X_3 + 1.102677 X_1 X_3^2 \\
- 1.964489.10^3 X_1 X_2^2 X_3 + 3.613635.10^4 X_1 X_2^2 \\
- 1.412911.10^5 X_1^2 X_3 + 1.931847.10^4 X_1^3 X_3 - 1.572845 X_1^3 X_2
\]

and its tuned version was obtained by multiplying by 0.9 the parameter corresponding to term \(X_3\). The 9-term model (of maximum degree \(q = 4\), also) reads:

\[
P(X_1, X_2, X_3) = -1.249580.10^1 X_3 - 7.266211.10^2 X_1 \\
- 3.141190X_1 X_3 + 4.819542.10^3 X_1 X_3^2 - 1.933067.10^4 X_1 X_2 X_3 \\
+ 1.415384X_1 X_2^2 - 1.259086.10^1 X_1^2 X_2 + 2.257435.10^4 X_1^3 X_3 \\
- 1.294766 X_1^3 X_2
\]

and the 10-term model (maximum degree \(q = 3\)) reads:

\[
P(X_1, X_2, X_3) = -1.671440.10^1 X_3 - 2.847528.10^4 X_2 X_3 + 1.481552 X_2^2 \\
- 9.143745.10^2 X_1 + 2.101548.10^3 X_1 X_3^2 - 3.109967.10^4 X_1 X_2 \\
+ 7.998291.10^4 X_1 X_2^2 + 1.237751 X_2^2 X_3 - 7.199582 X_2^3 X_2 \\
+ 5.430887.10^1 X_1^3
\]

The tuned model derived from this latter 10-term model was obtained by multiplying by 0.92 the parameter corresponding to term \(X_3\). The following (Markov) transfer matrix and corresponding binary transfer matrix were estimated from the first return maps shown in Figure 6 (see Tables 2 & 3 for details). For the 13-term model, these matrices read:

\[
\begin{align*}
0 & \quad 1 & \quad 2 & \quad 3 & \quad 0 & \quad 1 & \quad 2 & \quad 3 \\
(0.21 & \quad 0.35 & \quad 0.44 & \quad 0) & \quad (1 & \quad 1 & \quad 1 & \quad 0) \\
0 & \quad 1 & \quad 2 & \quad 3 & \quad 0 & \quad 1 & \quad 2 & \quad 3 \\
0 & \quad 0 & \quad 0.10 & \quad 0.12 & \quad 0 & \quad 0 & \quad 1 & \quad 1 & \quad 0 & \quad 0 & \quad 1 & \quad 1 & \quad 0 \\
1 & \quad 0.10 & \quad 0.11 & \quad 0.14 & \quad 0 & \quad 1 & \quad 1 & \quad 1 & \quad 1 & \quad 0 \\
2 & \quad 0.10 & \quad 0.13 & \quad 0.19 & \quad 0 & \quad 2 & \quad 1 & \quad 1 & \quad 1 & \quad 0 \\
3 & \quad 0 & \quad 0 & \quad 0 & \quad 0 & \quad 3 & \quad 0 & \quad 0 & \quad 0 & \quad 0
\end{align*}
\]

and after tuning the parameter corresponding to term \(X_3\):
For the 10-term model, transfer matrices read:

\[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0.15 & 0.40 & 0.45 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & 1 & 1 & 0
\end{pmatrix}
\]

(10)

and after tuning the coefficient corresponding to term \(X_3\):

\[
\begin{pmatrix}
0.26 & 0.29 & 0.20 & 0.25 \\
0.04 & 0.07 & 0.06 & 0.10
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 2 & 3 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]

(13)

These matrices should be compared to the matrices estimated from the original signal of variable \(x_1\):
The small differences, when comparing the matrices obtained from variable \( x_2 \) (with Eq. 15) with these latter ones obtained from variable \( x_1 \) (Eq. 24), may arise from the imprecision when choosing the boundary limit between two symbols in the partition of the first return map.

### 1.3 Rössler-\( x_3 \)

One 30-term model was obtained from the \( x_3 \) variable of the Rössler system. The model (of maximum degree \( q = 5 \)) reads:

\[
P(X_1, X_2, X_3) = 1.469183837 X_1 X_2 X_3 + 3.298435897 \cdot 10^7 X_2^2
- 5.079722432 \cdot 10^6 X_2^2 X_3^2 - 0.095501855 X_3^3
- 2.454817807 \cdot 10^4 X_2^3 X_3 - 0.000743108 X_2^4 - 8.516901345 X_1 X_3
- 0.007078151 X_1 X_3^2 + 3.822423364 \cdot 10^6 X_1 X_3^3
- 0.457986519 X_1 X_2 X_3 + 3.536469082 \cdot 10^5 X_1 X_2 X_3^2
- 1.222762045 \cdot 10^4 X_1 X_2^2 + 4.453556554 \cdot 10^7 X_1 X_2^2 X_3^2
+ 0.010954386 X_1 X_2^3 + 0.003155023 X_1 X_2^3
- 3.343660288 \cdot 10^7 X_1 X_2^3 X_3 - 3.328680543 \cdot 10^7 X_1 X_2^3 X_3
+ 0.065324718 X_1 X_2 X_3 - 1.635579935 \cdot 10^6 X_1 X_2 X_3^2
+ 1.294265505 X_1 X_2 X_3^2 + 0.000160434 X_1 X_2 X_3^3
- 0.001047247 X_1 X_2 X_3^2 - 6.374848605 \cdot 10^2 X_1 X_3^2 + 0.876100551 X_1 X_3^2
- 0.000320012 X_1 X_1 X_3 + 4.724411452 X_1 X_3
- 0.001967624 X_1 X_2 X_3 + 2.265117133 \cdot 10^3 X_1^4
- 0.122449442 X_1 X_3 - 2.165857333 \cdot 10^1 X_1^5
\]

The tuned version of the model was obtained by multiplying by 1.002 the parameter corresponding to term \( X_1^5 \). The following (Markov) transfer matrix and corresponding binary transfer matrix were estimated from the first return maps shown in Figure 7 (see Tables 2 & 3 for details). For the 30-term (not tuned) model, these matrices read:

\[
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0.22 & 0.29 & 0.23 & 0.25 & 1 & 1 & 1 & 1
\end{pmatrix}
\]
and after tuning the coefficient of term $X_2^2$, transfer matrices become:

$$
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0.02 & 0.16 & 0.01 & 0 \\
0.17 & 0.31 & 0.16 & 0 \\
0.01 & 0.15 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0.02 & 0.16 & 0.01 & 0 \\
0.17 & 0.31 & 0.16 & 0 \\
0.01 & 0.15 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

These distributions and matrices should be compared to the transfer matrices estimated from the original variable $x_3$:

$$
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0.08 & 0.02 & 0.05 & 0.05 \\
0.03 & 0.05 & 0.03 & 0.06 \\
0.04 & 0.10 & 0.05 & 0.08 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 2 & 3 \\
0.08 & 0.02 & 0.05 & 0.05 \\
0.03 & 0.05 & 0.03 & 0.06 \\
0.04 & 0.10 & 0.05 & 0.08 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

Here again, small differences (when comparing to the matrices resulting from variables $x_2$ and $x_1$, see Eq. 15 & 24) may arise from the imprecision when choosing the boundary limits for the symbols’ partition in the first return map.

2 The electrodissolution of copper in phosphoric acid

Two parameterizations of the 21-term model (of maximum degree $q = 4$) were obtained for the electrodissolution of copper in phosphoric acid. The first one (noted 21-p., see figures 8c and 8d) reads:
\[ P(X_1, X_2, X_3) = 4.376 \times 10^3 + 2.540 \times 10^4 X_3^2 - 3.502 \times 10^5 X_3^4 \\
+ 1.088 \times 10^6 X_3^4 + 6.370 \times 10^6 X_2 X_3 - 4.782 \times 10^4 X_2 X_3^2 \\
+ 1.522 \times 10^5 X_3 X_3^4 + 8.113 X_3^6 X_3^2 - 1.198 \times 235.10^2 X_1 X_3^2 \\
+ 8.081 \times 10^2 X_1 X_3^3 + 1.541 \times 918.10^3 X_1 X_2 + 9.296 \times 10^8 X_1 X_2 X_3^2 \\
- 6.117 \times 464.10^5 X_1 X_2^2 X_3 - 5.280 \times 800.10^5 X_1 X_3^3 - 5.282 \times 890.10^3 X_1^2 X_3 \\
+ 1.415 \times 405.10^2 X_1^2 X_3^2 - 1.762 \times 10^4 X_1^2 X_2 - 1.343 \times 480.10^2 X_1^2 X_2 X_3 \\
- 4.303 \times 322.10^3 X_1^2 X_2^2 - 5.855 \times 790.10^3 X_1^3 - 4.677 \times 767.10^3 X_1^3 X_2 \\
\]  

and the second one (noted 21-p.\* \( \text{opt} \), see Figure 8e and 8f) reads:

\[ P(X_1, X_2, X_3) = 3.487 \times 242.10^3 + 3.556419 \times 10^4 X_3^2 - 5.142 \times 267.10^5 X_3^4 \\
+ 1.817 \times 063 \times 10^6 X_3^4 + 5.612 \times 236 \times 10^4 X_2 X_3 - 4.13546 \times 10^4 X_2 X_3^2 \\
+ 1.349334 \times 10^5 X_2 X_3^3 + 6.591543 X_2^2 - 1.628945 \times 10^2 X_1 X_3^2 \\
+ 1.171202 \times 10^5 X_1 X_3^3 + 6.503816 X_1 X_2 + 7.907225 \times 10^5 X_1 X_2 X_3^2 \\
- 5.248 \times 275 \times 10^5 X_1 X_2^2 X_3 - 4.267778 \times 10^4 X_1 X_3^3 - 4.468080 \times 10^3 X_1^2 X_3 \\
+ 1.878962 \times 10^4 X_1^2 X_3 - 1.260730 \times 10^4 X_1^2 X_2 - 1.183509 \times 10^4 X_1^2 X_2 X_3 \\
- 3.671600 \times 10^3 X_1^2 X_2^2 - 4.723065 \times 10^2 X_1^3 - 6.952164 \times 10^3 X_1^3 X_2 \\
\]  

The tuned version of this second model (noted 21-p.\* \( \text{opt} \)) was obtained by multiplying by 1.002 the parameter corresponding to term \( X_1^3 X_2 \) (see Figure 8g). The (Markov) transfer matrix and corresponding binary transfer matrix were estimated from the first return map shown in Figure 8, (see Tables 2 & 3 for details). For the 21-term model (21-p.) \( \text{p} \), these matrices read:

\[
\begin{pmatrix}
0 & 1 & 0 & 1 \\
0.30 & 0.70 & \text{(1, 1)} & \\
0 & 1 & 0 & 1
\end{pmatrix}
\]

(21)

And for the alternative 21-term model, the following matrices were obtained using the tuned version of the model (21-p.\* \( \text{opt} \)):

\[
\begin{pmatrix}
0 & 1 & 0 & 1 \\
0.44 & 0.55 & \text{(1, 1)} & \\
0 & 1 & 0 & 1
\end{pmatrix}
\]

(22)

These distributions and matrices should be compared to the estimates obtained from the original data set:
The cycle of rainfed wheat

One 15-term model was obtained for rainfed wheat in the semi-arid climatic region in North Morocco. The model is of maximum degree 3 \((q = 3)\) and reads:

\[
P(X_1, X_2, X_3) = -1.309 \times 10^3 - 4.504 \times 10^3 X_2 \\
+ 8.897 \times 10^4 X_3 X_1 X_3 - 6.763 \times 10^4 X_2 X_3^2 - 2.391 \times 10^5 X_2^2 \\
- 1.597 \times 10^5 X_3^3 + 9.416 \times 10^4 X_1 - 8.571 \times 10^2 X_1 X_3 \\
+ 2.064 \times 10^4 X_1 X_2 - 1.926 \times 10^2 X_1 X_2 X_3 \\
+ 4.843 \times 10^2 X_1 X_2^2 - 2.163 \times 10^3 X_1^2 + 1.824 \times 10^2 X_1^2 X_3 \\
- 2.331 \times 10^4 X_1^2 X_2 + 1.604 \times 10^3 X_1^3
\]

3 References